Math

- **Linear Algebra**
  - **Vector**
    - **Defi**
      - Vector v written as **v** or $\vec{v}$.
      - A vector from (0, 0, … , 0) pointing to (v_1, v_2, … , v_n) is written as $\vec{v} := (v_1, v_2, \ldots, v_n)$.
      - For measuring direction and magnitude
    - **Basic Operations**

      For all vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ and scalars a, b:

      - $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
      - $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$
      - There exists a *zero vector* "$\mathbf{0}$" such that $\mathbf{v} + \mathbf{0} = \mathbf{0} + \mathbf{v} = \mathbf{v}$
      - For every $\mathbf{v}$ there is a vector "$-\mathbf{v}$" such that $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$
      - $1\mathbf{v} = \mathbf{v}$
      - $a(b\mathbf{v}) = (ab)\mathbf{v}$
      - $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$
      - $(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}$
    - **Inner Product**
      - $\langle \mathbf{u}, \mathbf{v} \rangle := \sum_{i=1}^{n} u_i v_i.$
      - properties
        - $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$
        - $\langle \mathbf{u}, \mathbf{u} \rangle \geq 0$, with equality only for $\mathbf{u} = \mathbf{0}$
        - $\langle a\mathbf{u} + b\mathbf{v}, \mathbf{w} \rangle = a\langle \mathbf{u}, \mathbf{w} \rangle + b\langle \mathbf{v}, \mathbf{w} \rangle$
    - **Norm**
      - $||\mathbf{u}|| := \sqrt{\langle \mathbf{u}, \mathbf{u} \rangle}.$
      - properties
        - $||\mathbf{u}|| \geq 0$, with equality only for $\mathbf{u} = \mathbf{0}$
        - $||\mathbf{u}|| = 0$ if and only if $\mathbf{u} = \mathbf{0}$
        - $||a\mathbf{u}|| = |a|||\mathbf{u}||$
        - $||\mathbf{u} + \mathbf{v}|| \leq ||\mathbf{u}|| + ||\mathbf{v}||$
  - **Linear Map**

    A map f between vector spaces is *linear* if

    $$f(\mathbf{x} + \mathbf{y}) = f(\mathbf{x}) + f(\mathbf{y})$$

    and

    $$f(a\mathbf{x}) = af(\mathbf{x}),$$
  - **Affine Map**

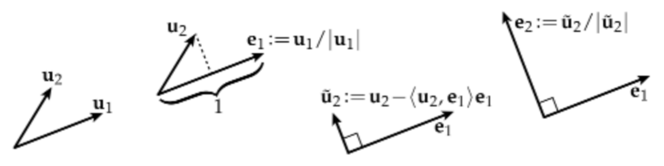
    if we have some set of weights $w_1, \ldots, w_n$ such that $\sum_{i=1}^{n} w_i = 1$, then a map f is affine if

    $$f(w_1\mathbf{x}_1 + \cdots + w_n\mathbf{x}_n) = \sum_i w_i f(\mathbf{x}_i)$$

    for any collection of vectors $\mathbf{x}_1, \ldots, \mathbf{x}_n$.
  - **Basis and Span**
    - The *span* of a collection of vectors $\mathbf{e}_1, \ldots, \mathbf{e}_n$ is the set of all vectors $\mathbf{u}$ that can be expressed as

      $$\mathbf{u} = u_1\mathbf{e}_1 + \cdots + u_n\mathbf{e}_n = \sum_{i=1}^{n} u_i \mathbf{e}_i$$

      for some set of coefficients $u_1, \ldots, u_m \in \mathbb{R}$. If the vectors $\mathbf{e}_1, \ldots, \mathbf{e}_n$ span all the vectors in $\mathbb{R}^n$, we say that the collection $\{\mathbf{e}_i\}$ is a *basis* for $\mathbb{R}^n$. If, in addition, $\langle \mathbf{e}_i, \mathbf{e}_i \rangle = 1$ for all i, and $\langle \mathbf{e}_i, \mathbf{e}_j \rangle = 0$ for $i \neq j$, we say that $\{\mathbf{e}_i\}$ is an *orthonormal basis*. Notice, by the way, that this definition depends on our choice of inner product $\langle \cdot, \cdot \rangle$.
    - **How to get Orthonormal Basis** — **Gram-Schmidt procedure**
      1. normalize the first vector (i.e., divide by its length)
      2. subtract any component of the 1st vector from the 2nd one
         > subtracting off any part of the previous vector is not orthogonal to the new vectors; only keeping the orthogonal part.
      3. normalize the 2nd one
      4. repeat, removing components of first k vectors from vector k+1

      > $u_n = u_n - < u_n, e_1 > e_1 - < u_n, e_2 > e_2 - \ldots - < u_n, u_{n-1} > e_{n-1},$
      >
      > where u_n stands for the n-th orthonormal basis.
  - **Systems of Linear Equations**

    Given some set of variables $x_1, \ldots, x_k \in \mathbb{R}$, a *(real) linear equation* is any equation of the form

    $$f(x_1, \ldots, x_k) = b$$

    where f is a linear function and $b \in \mathbb{R}$ is a constant.

    A *system* of linear equations is simply a collection of linear equations

    $$\begin{aligned} f_1(x_1, \ldots, x_k) &= b_1 \\ &\vdots \\ f_p(x_1, \ldots, x_k) &= b_p, \end{aligned}$$

    each of which shares the same set of variables. Solving a linear system means finding values for the variables $x_1, \ldots, x_k$ that satisfy all of the equations simultaneously. We will also sometimes refer to these variables as *degrees of freedom*.
  - **Matrices**
    - *Especially in computer graphics, matrices almost always have a very concrete geometric meaning — e.g., a rotation, a re-scaling, an energy, etc.
- **Vector Calculus**
  - **Dot Product**

    In our study of linear algebra, we talked about *inner products* abstractly, *i.e.*, we said that an inner product $\langle \cdot, \cdot \rangle$ was *any* operation that is symmetric, bilinear, *etc.* When working with two- and three-dimensional geometry, we typically want to work with one very special inner product called the **Euclidean inner product**, which has a concrete geometric relationship to lengths and angles. In particular, for any two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, the Euclidean inner product is defined as

    $$\langle \mathbf{u}, \mathbf{v} \rangle_{\text{Euc}} := |\mathbf{u}||\mathbf{v}|\cos(\theta).$$

    where $|\mathbf{u}|$ and $|\mathbf{v}|$ are the lengths of $\mathbf{u}$ and $\mathbf{v}$, respectively, and $\theta \geq 0$ is the (unsigned) angle between them. If the components u_i, v_i of these two vectors are expressed with respect to some orthonormal basis $\mathbf{e}_1, \ldots, \mathbf{e}_n$, then the Euclidean inner product can be computed via the **dot product**

    $$\mathbf{u} \cdot \mathbf{v} := \sum_{i=1}^{n} u_i v_i,$$

    Notice that we made two key assumptions here:

    1. The vectors $\mathbf{u}, \mathbf{v}$ represent vectors in $\mathbb{R}^n$; they are not tangent vectors on the sphere, bitmap images, or rows in a database.
    2. The values $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ are the components of $\mathbf{u}$ and $\mathbf{v}$ with respect to an *orthonormal basis*.

    If either of these two assumptions are violated, then the dot product no longer carries the geometric meaning one might expect, *i.e.*, it is no longer true that $\mathbf{u} \cdot \mathbf{v} = \langle \mathbf{u}, \mathbf{v} \rangle_{\text{Euc}}$. If we want to recover length or angle in a different basis, we need to carefully account for the effect of our choice of basis on the coordinates—mismanagement of coordinate systems is common source of bugs in graphics code.
  - **Cross Product**

    Unlike the dot product, which maps two vectors to a scalar, the **cross product** maps two vectors to another vector. In particular, for two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$, the cross product can be defined as the unique vector $\mathbf{u} \times \mathbf{v} \in \mathbb{R}^3$ such that

    $$\sqrt{\det(\mathbf{u}, \mathbf{v}, \mathbf{u} \times \mathbf{v})} = |\mathbf{u}||\mathbf{v}|\sin(\theta)$$

    where det denotes the determinant of a matrix, and $\theta \in [0, \pi]$ is the angle between $\mathbf{u}$ and $\mathbf{v}$.[2] From these three properties, one can infer that in an orthonormal coordinate system the cross product must be equal to

    $$\mathbf{u} \times \mathbf{v} = \begin{bmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{bmatrix}.$$

    Note that this operation is only well-defined for vectors in $\mathbb{R}^3$. However, it is sometimes convenient (especially in computer graphics) to abuse notation and write

    $$\mathbf{u} \times \mathbf{v} := u_1 v_2 - u_2 v_1$$

    for a pair of vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2$. This expression effectively treats $\mathbf{u}$ and $\mathbf{v}$ as 3-vectors $(u_1, u_2, 0)$ and $(v_1, v_2, 0)$, yielding the third (nonzero) component as the result. The dot and cross product are extremely important in many computer graphics concepts, so we highly encourage you to spend some time thinking about their geometric meaning.
  - **Derivatives & Integration**